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SEMIINVARIANTS FOR HOPF ALGEBRA ACTIONS

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ABSTRACT

Let H be a Hopf algebra and M a representation or a corepresentation of H. In this paper we study semiinvariants of M. This notion generalizes the known concept of weight spaces in the context of representations of Lie algebras. Our best results are attained for pointed Hopf algebras, and semiinvariants which are related to the coradical filtration of H.

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Introduction

Let H be a finite-dimensional Hopf algebra over a field k, and M a left H-module. Let $G = G(H^*)$, the group of group-likes of H^*. For each $\sigma \in G$ let $M_\sigma = \{a \in M | \, h \cdot a = \langle \sigma, h \rangle a$, for all $h \in H\}$, if $M_\sigma \neq 0$ then σ is called a weight for M with weight space M_σ, and $M_G = \sum_{\sigma \in G} M_\sigma$ is a $(kG)^*$-module, that is, a G-graded space. If $M = A$, an H-module algebra, then A_G is a subalgebra of A and is a $(kG)^*$-module algebra. This subalgebra was introduced in [BCM] where it was termed the subalgebra of semiinvariants, A_G, generalizing a known notion for groups acting by automorphisms, or Lie algebras acting by derivations.

A dual way of viewing M_σ is by considering ρ, the induced coaction of H^* on A, then:

$$M_\sigma = \rho^{-1}(M \otimes \sigma) \quad \text{and} \quad M_G = \rho^{-1}(M \otimes kG).$$

This dual view suggests the notion which we term K-semiinvariants, as follows: Let H be a Hopf algebra and K a subcoalgebra of H, let M be an H-comodule then the K-semiinvariants of M are:

$$M_K = \rho^{-1}(M \otimes K), \text{ a subcomodule of } M.$$

In this paper we study K-semiinvariants with special emphasis on the chain of the subcoalgebras $C_i = \wedge^i kG$, where $G = G(H)$, and for which M_G is just the first term in a chain of H-subcomodules. Our results are sharpest for pointed Hopf algebras, that is, Hopf algebras for which all simple subcoalgebras are 1-dimensional. In this situation, $\{C_i\}$ forms a filtration of H (termed the coradical filtration), and M_{C_i} form a comodule filtration of M. Pointed Hopf algebras occur in abundance: Group algebras, or more generally any cocommutative Hopf algebras over algebraically closed fields, enveloping algebras of Lie algebras and their quantizations $[M, R]$, Sweedler's four-dimensional Hopf algebra (0.5), are all examples of such algebras.

In Theorem 1.4 we prove that if H is pointed and A is an H-comodule algebra so that there exists a total integral φ: $H \to A$, then A/A^{coH} is H-Galois (H-cleft) if and only if A_G is strongly graded (A_G/A^{coH} is kG-cleft).

On the other hand, in Theorem 3.2 we show that if H is a finite-dimensional pointed Hopf algebra and A is an H-module algebra, then $A \# H$ is a finite subnormalizing extension of A. This yields a strong connection between $A -$ Mod and $A \# H -$ Mod.

On studying the chain M_{C_i} we put special emphasis on the first two terms $M_G = M_{C_0}$ and M_{C_1}. The importance of C_1 lies in the fact that it contains all the nearly (skew) primitive elements, these elements give rise to derivations and skew-derivations. We furthermore interrelate these comodule semiinvariants with module actions. This is especially emphasized when H is finite-dimensional.

The paper is organized as follows:

In section 1 we study semiinvariants of coactions under the assumption that there exists a total integral (which in the finite-dimensional case is equivalent to the existence of an element of trace 1 under the action of the dual). Such a total integral induces maps from A to A_K, which for $K = k\sigma$ are projections (denoted by π_σ) onto A_σ. Using these projections we prove in Theorem 1.3 that there is a strong connection between A being H-Galois and A_G being strongly graded. This culminates in 1.4 for pointed H.

In section 2 we study the original point of view, that of semiinvariants of actions. For finite dimensional H we use here a combination of H-module or H^*-comodule theoretic points of view. Of special interest are the elements $t^\sigma \in H$, where t is a left integral, $\sigma \in G(H^*)$ and $t^\sigma = \sigma \rightharpoonup t$. When A has an element of trace 1 these are the analogues of the maps π_σ of section 1. We give explicit module-formulations for M_σ, and $M_{k\tau \wedge k\sigma}$ using these ideas. In Theorem 2.7 we use 1.3 when A/A^H is H^*-Galois and show that more can be said in the finite-dimensional case. For example, $A_\sigma \neq 0$ for each $\sigma \in G$, and if H is semisimple then $A_G \#(kG)^*$ is isomorphic as an algebra to an algebra $S \subset A\#H$. In Theorem 2.8 we carry this further and assume that A is an irreducible $A\#H$-module (this is true of course for A =division ring). Here we show that A_G is a crossed product, and if $[A : A^H] = \dim H$ then $A_G = A^H \#_\mu kG$.

In section 3 we continue as in section 2 to study finite-dimensional H acting on A, and our interest lies in the categories of $R -$ Mod where $R = A, A^H, A_G$ or $A\#H$, and their interrelations. When H is pointed there is an especially strong connection as a consequence of $A\#H/A$ being a finite subnormalizing extension. This is summarized in 3.3–3.6. In a somewhat different direction we use categorical properties of the Morita context which arise for H-actions [as in CFM] to relax some of the conditions in Theorem 2.7. This is summarized in 3.10–3.12, where we show various equivalences with quotient categories.

0. Preliminaries

Let H be a Hopf algebra over a field k with comultiplication Δ, counit ε and antipode S. We use Sweedler's [S] "sigma" notation for Δ; that is: $\Delta(h) = \sum h_1 \otimes h_2$, all $h \in H$. Let A be an H-module algebra, (we also say here that H acts on A). Form the smash product $A\#H$ which is $A \otimes H$ as a vector space, and multiplication is defined by: $(a\#h)(b\#g) = \sum a(h_1 \cdot b)\#h_2 g$, all $a, b \in A$, $h, g \in H$. Since A and H are embedded as algebras in $A\#H$ we shall often write ah for $a\#h$.

Another related algebra is the subalgebra A^H of H-invariants defined by

$$A^H = \{a \in A\colon h \cdot a = \varepsilon(h)a, \quad \text{all } h \in H\}.$$

Moreover, just as in the case of automorphisms and derivations,a larger subalgebra of A, A_G, called the "semiinvariant" subalgebra, was introduced in [BCM]. Let

$$G(H^*) = \{\text{algebra homomorphisms from} H \text{to} k\} = \{\sigma \in H^* \mid \Delta(\sigma) = \sigma \otimes \sigma\}.$$

Definition 0.1: Let M be a left H-module (or H-module algebra) and let $\sigma \in G(H^*)$. Set

$$M_\sigma = \{a \in A\colon h \cdot a = \langle \sigma, h\rangle a, \text{ all } h \in H\}.$$

If $M_\sigma \neq 0$ then we call M_σ the weight space of M with respect to σ and so if G is finite then $M_G = \sum_\sigma M_\sigma$ is a $(kG)^*$-module (module algebra resp.). Moreover, if $M = A$ is an H-module algebra then we can form $A_G\#(kG)^*$ [CM] (usually unrelated to $A\#H$). Note that $A^H = A_\varepsilon \subset A_G$.

When H is finite-dimensional there is a dual way of viewing M_σ by considering the induced H^*-comodule (coaction) structure on M, which is given by:

$$\rho\colon M \to M \otimes H^* \text{ via } a \to \sum h_i \cdot a \otimes h_i^*, \quad a \in M,$$

where $\{h_i\}$, $\{h_i^*\}$ are dual bases of H and H^* respectively. Conversely if M is an H^*-comodule (algebra) with $\rho\colon M \to M \otimes H^*$ denoted by $\rho(a) = \sum a_o \otimes a_1$, then ρ induces a left H-module (action) structure on M by:

$$h \cdot a = \sum \langle a_1, h\rangle a_0.$$

Under the above it is straightforward to show that

$$M_\sigma = \rho^{-1}(M \otimes \sigma) \quad \text{and} \quad M_G = \rho^{-1}(M \otimes kG).$$

Taking the dual view we shall consider M an H-comodule (algebra) and for certain subcoalgebras K of H the K-semiinvariants: $M_K = \rho^{-1}(M \otimes K)$. Specifically, let us recall, that if I, J are subcoalgebras of H, then $I \wedge J \equiv \Delta^{-1}(I \otimes H + H \otimes J)$ is a subcoalgebra of H containing both I and J. For $I = kG$, where $G = G(H) = \{g \in H | \, \Delta(g) = g \otimes g\}$ (the so-called group-like elements of H), setting $C_0 = kG$, $C_i = \wedge^i kG = (\wedge^{i-1} kG) \wedge kG$ we get a chain of subcoalgebras:

$$k \subset C_0 \subset C_1 \cdots$$

This chain is part of the coradical filtration of H [S], and contains some fundamentally important elements. First a definition:

Definition 0.2: Let C be a coalgebra over k and $G = G(C)$. Let $\sigma, \tau \in G$ then $x \in C$ is said to be σ: τ nearly primitive if

$$\Delta(x) = x \otimes \sigma + \tau \otimes x$$

(this relation implies that $\varepsilon(x) = 0$). ∎

Denote by $P_{\sigma,\tau}(C) = \{\sigma\colon \tau \text{ nearly primitive elements of } C\}$. These elements play a central role in the study of certain quantum groups (see Examples 0.4 and 0.5). When $C = H$ is a Hopf algebra then the primitive elements of H are the $1:1$ nearly primitives and the so-called σ-primitives are the $\sigma:1$ nearly primitive elements. When H acts on A such elements give rise to derivations, or σ-derivations respectively [KP,MS].

By definition, $P_{\sigma,\tau} \subset C_1$, so analogously, for any i, let $x \in C$ satisfy:

$$(*) \qquad \Delta(x) - x \otimes \sigma - \tau \otimes x \in C_{i-1} \otimes C_{i-1}, \qquad \text{some } \sigma, \tau \in G(C);$$

then $x \in C_i$ by definition.

These elements will be of central importance in this paper as well. To see this recall that a coalgebra is pointed if and only if its coradical equals kG, and then by [TW], if K_i is any vector space complement of C_{i-1} in C_i then $K_i = \sum_{\sigma,\tau \in G} K_{i,\sigma,\tau}$ where $K_{i,\sigma,\tau} = \{x \in K_i \text{ satisfying } (*)\}$. Since the elements of $K_{i,\sigma,\tau}$ are "nearly-primitive mod C_{i-1}", the above allows for using induction. We list the following which are either known or proved by induction.

LEMMA 0.3: *Let H be a Hopf algebra, $C_i = \bigwedge^i kG$, $i \geq 0$ and $C = \bigcup_{i\geq 0} C_i$ then*

(1) *C_i is S-stable.*
(2) *$C_iC_j \subset C_{i+j}$, for all $i, j \geq 0$* [M,1.1].
(3) *C is a pointed sub-Hopf algebra of H* [S,11.1.1] *with bijective antipode (by* [T]*).*
(4) *$\langle C_i \rangle$, the subalgebra of H generated by C_i is a pointed sub-Hopf algebra of C.*

Some examples of pointed Hopf algebras:

Example 0.4: [M] The Quantum group $H = U_q((2))$.

It is $k[E, F, K, K^{-1}]$ defined by the relations: $KE = q^2EK$, $KF = q^{-2}FK$, $EF - FE = \frac{K^2-K^{-2}}{q^2-q^{-2}}$. It is a Hopf algebra by:

$$\begin{aligned}\Delta(E) =& E \otimes K^{-1} + K \otimes E,\\ \Delta(F) =& F \otimes K^{-1} + K \otimes F,\\ \Delta(K) =& K \otimes K.\end{aligned}$$

Hence, $K \in G(H)$ and $E, F \in P_{K^{-1},K}(H)$. Similarly the Hopf algebra $H = U_q(g)$, where g is a Kac–Moody Lie algebra of finite or affine type over $\mathbb{C}$, is generated by $\{k_i^{\pm 1}, e_i, f_i\}$ with certain relations concerning its multiplicative structure, where $k_i \in G(H)$, and $\{e_i, f_i\} \subset P_{k_i{}^{-1},k_i}(H)$.

Example 0.5: Sweedler's 4-dimensional (triangular) Hopf algebra, which is $H = k[g, x]$ with relations: $g^2 = 1$, $x^2 = 0$ and $xg = -gx$. It is a Hopf algebra by: $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$. Here $g \in G(H)$, $x \in P_{g,1}(H)$, and by 0.3 H is pointed.

We also list some properties of $P_{\sigma,\tau}(C)$:

LEMMA 0.6: *Let C be a coalgebra, and $\sigma, \tau \in G(C)$ then:*

(1) $P_{\sigma,\tau}(C) = (k\tau \wedge k\sigma) \cap \operatorname{Ker} \varepsilon$.
(2) $\sigma - \tau \in P_{\sigma,\tau}(C)$.
(3) *If C is cosemisimple then $P_{\sigma,\tau}(C) = k(\sigma - \tau)$, in particular there are no primitive elements in C.*
(4) $k\tau \wedge k\sigma = P_{\sigma,\tau}(C) \oplus k\tau$.

If moreover $C = H$ is a Hopf algebra then:

(5) *If $\mu \in G(H)$, then $\mu P_{\sigma,\tau}(H) = P_{\mu\sigma,\mu\tau}(H)$.*

(6) $S(k\tau \wedge k\sigma) = k\sigma^{-1} \wedge k\tau^{-1}$, *thus in particular*

$$P_{\tau^{-1},\sigma^{-1}}(H) = \tau^{-1}P_{\sigma,\tau}(H)\sigma^{-1} = S(P_{\sigma,\tau}(H)).$$

Proof: We only prove (1) by generalizing [S, 10.0.1]. The rest are known or can be deduced directly or from [TW] or (1). Let $x \in (k\tau \wedge k\sigma) \cap \operatorname{Ker}\varepsilon$, then $\Delta(x) = z \otimes \sigma + \tau \otimes y$, some $y, z \in C$. Applying $\mathrm{id} \otimes \varepsilon$ and $\varepsilon \otimes \mathrm{id}$ to this expression we get: $\tau\varepsilon(y) + z = x = y + \varepsilon(z)\sigma$ and hence $z = x - \varepsilon(y)\tau$, and $y = x - \varepsilon(z)\sigma$. Since $\varepsilon(x) = 0$ we also get $\varepsilon(z) + \varepsilon(y) = 0$. Thus,

$$\Delta(x) = \tau \otimes y + z \otimes \sigma = \tau \otimes (x - \varepsilon(z)\sigma) + (x - \varepsilon(y)\tau) \otimes \sigma$$

and hence, since $\varepsilon(z) + \varepsilon(y) = 0$, we get $\Delta(x) = \tau \otimes x + x \otimes \sigma$ as claimed.

The following example indicates that cosemisimplicity of H in (3) is essential:

Example 0.7: Let k be a field of characteristic 2 and $G = \langle\sigma\rangle$ a cyclic group of order 2. Let $H = (kG)^*$ (with basis p_1, p_σ), then H is not cosemisimple and indeed p_σ is a primitive element for

$$\begin{aligned} 1 \otimes p_\sigma + p_\sigma \otimes 1 &= (p_1 + p_\sigma) \otimes p_\sigma + p_\sigma \otimes (p_1 \otimes p_\sigma) \\ &= p_1 \otimes p_\sigma + p_\sigma \otimes p_1 = \Delta(p_\sigma). \end{aligned}$$

The main results of our paper are concerned with A, an H-comodule algebra, where the coaction has special properties. Let $A^{\mathrm{co}H} = \rho^{-1}(A \otimes 1)$, then the extension $A/A^{\mathrm{co}H}$ is said to be H-**Galois** [KT] if

$$\beta\colon A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_k H$$

given by $a \otimes b \mapsto (a \otimes 1)\rho(b)$ is a bijection. One may also consider the map $\beta'\colon A \otimes_{A^{\mathrm{co}H}} A \to A \otimes H$ given by $a \otimes b \mapsto \rho(a)(b \otimes 1)$. The same proof as [KT, 1.2] implies that when H has a bijective antipode then β is bijective if and only if β' is bijective. Hence either definition is used in this paper. When H is finite dimensional and A is an H-module algebra (hence an H^*-comodule algebra) then $A^{\mathrm{co}H^*} = A^H$ and A/A^H is H^*-Galois iff the map $[\ ,\ ]\colon A \otimes_{A^H} A \to A\#H$, given by $a \otimes b \mapsto (a\#t)(b\#1)$ where t is a left integral of H, is surjective [CFM].

In a somewhat different direction, if A is an H-comodule algebra we say that there exists a **total integral** [Do1] if there exists a comodule map $\varphi\colon H \to A$ so that $\varphi(1) = 1$. When H is finite-dimensional and A is an H-module algebra then by [CF] there exists a total integral φ: $H^* \to A$ iff there exists $c \in A$ so that $t \cdot c = 1$ (c is called an **element of trace** 1). If H is semisimple then 1_A is such an element. It was shown in [Do1] that the existence of a total intergal φ: $H \to A$ is equivalent to A being an injective H-comodule. That is, for an injective H-comodule map i: $U \to V$ and any comodule map f: $U \to A$ there exists an H-comodule map g: $V \to A$ so that $gi = f$.

Moreover, if there exists a right H-comodule map φ: $H \to A$ which is invertible in the convolution algebra $\mathrm{Hom}(H, A)$ then A is called H-**cleft** over $A^{\mathrm{co}H}$. By [BM] this is equivalent to A being a crossed product $A^{\mathrm{co}H} \#_\sigma H$.

When H is finite-dimensional and A is an H-module algebra then the above define a Morita context [CFM]:

$$[A^H,\ {}_{A^H}A_{A\#H},\ {}_{A\#H}A_{A^H},\ A\#H]$$

with defining maps: [,] as above, and (,): $A \otimes_{A\#H} A \to A^H$ given by $a \otimes b \to t \cdot (ab)$. Hence if both maps are surjective (that is, A/A^H is H^*-Galois and there exists an element of trace 1) then A^H and $A\#H$ are Morita equivalent.

Example 0.8: A well known example of an action for which both maps are surjective is the action of H^* on H (where H is finite-dimensional) given by:

$$p \rightharpoonup h = \sum \langle p, h_2 \rangle h_1$$

for all $p \in H^*$, $h \in H$.

The fact that the map [,] is surjective follows from [Do2], and the fact that (,) is surjective follows since H is Frobenius, which implies that $T \rightharpoonup t = 1$, where T and t are left integrals of H^* and H respectively.

Example 0.9: Let G be a group, and A a kG-comodule algebra, that is A is G-graded and $A^{\mathrm{co}kG} = A_1$. By [U], A/A_1 is kG-Galois if and only if $A_g A_{g^{-1}} = A_1$, for all $g \in G$. (A is then said to be strongly graded by G.) For finite G, being G-graded is equivalent to being a $(kG)^*$-module algebra [CM]. Since $(kG)^*$ is always semisimple, 1_A is an element of trace 1.

Finally, we recall from [C] and [CFM]; if A is an H-module algebra then, for all $h \in H$, $a, b \in A$

$$(h \cdot a)b = \sum h_1 \cdot (aS(h_2) \cdot b). \qquad (0.10)$$

If S is bijective, then

$$a(h \cdot b) = \sum h_2 \cdot (S^{-1}(h_1) \cdot a)b. \tag{0.11}$$

$$ah = \sum h_2(S^{-1}(h_1) \cdot a). \tag{0.12}$$

1. Semiinvariants of coactions

Let H be a Hopf algebra over k, $G = G(H)$ and $C_i = \wedge^i kG$. Let A be an H-comodule algebra and $A_K = \rho^{-1}(A \otimes K)$ the K-semiinvariants of A where K is a subcoalgebra of H. In this section we study the chain of semiinvariants

$$A_G = A_{C0} \subset A_{C1} \subset \cdots$$

which for pointed Hopf algebras yields a filtration of A. We prove that in the presence of a total integral there are strong connections between the extensions $A_G/A^{\mathrm{co}H}$, $A_{\langle C_i\rangle}/A^{\mathrm{co}H}$ and $A/A^{\mathrm{co}H}$ which culminate in Theorem 1.3, one of the main theorems of this paper.

The existence of a total integral gives rise to maps $A \to A_K$ which are projections for $K = k\sigma$, $\sigma \in G$. We discuss these maps in the following lemma.

Lemma 1.1: *Let φ: $H \to A$ be a total integral. Let $h \in H$ and define π_h: $A \to A$ by*

$$\pi_h(a) = \sum a_0\varphi(S(a_1)h), \qquad \textit{for all } a \in A.$$

Then,

(1) $\rho(\pi_h(a)) = \sum_{(h)} \pi_{h_1}(a) \otimes h_2$.
(2) $\pi_h(1) = \varphi(h)$.
(3) $a = \sum_{(a)} \pi_{a_1}(a_0)$.
(4) *If K is a subcoalgebra of H then*

$$A_K = \sum_{h\in K} \pi_h(A).$$

(5) *Let $\sigma \in G$ and $K = k\sigma$; then π_σ: $A \to A_\sigma$ is a projection (of $A^{\mathrm{co}H}$-modules).*
(6) *If H has a bijective antipode then $\pi'_\sigma(a) = \sum \varphi(\sigma S^{-1}a_1)a_0$ satisfies (5).*

Proof: (1) $\rho(\pi_h(a)) = \sum(a_0 \otimes a_1)\rho\varphi(S(a_2)h)$. However, $\rho\varphi = (\varphi \otimes \mathrm{id})\Delta$ hence this equals

$$\sum(a_0 \otimes a_1)[\varphi(S(a_3)h_1) \otimes S(a_2)h_2] = \sum a_0\varphi(S(a_1)h_1) \otimes h_2$$
$$= \sum \pi_{h_1}(a) \otimes h_2.$$

(2) follows from the definition of π_h.

(3) Let $a \in A$, then $a = \sum a_0\varepsilon(a_1) = a_0\varphi(\varepsilon(a_1)) = \sum a_0\varphi(S(a_1)a_2) = \sum \pi_{a_1}(a_0)$.

(4) The inclusion $\pi_h(A) \subset A_K$, for all $h \in K$, follows from (1) and the fact that K is a subcoalgebra. The converse follows from (3).

(5) Restricting to $K = k\sigma$ we have by (3) that for $a \in A_\sigma$, $a = \pi_\sigma(a)$, and by (4) that $\pi_\sigma(A) \subset A_\sigma$. Hence π_σ is a projection onto A_σ.

(6) The proof is similar to the above.

In the following proposition we study the chain of subcoalgebras C_i and we use Lemma 1.1 to show that although C_i and A_{C_i}, unlike C_0 and A_{C_0}, are not necessarily subalgebras of H and A respectively, this can be improved in the presence of a total integral.

Denote by $\langle X\rangle$ the subalgebra generated by a subset X.

PROPOSITION 1.2: *Let H be a Hopf algebra, A an H-comodule algebra and assume there exists a total integral φ: $H \to A$, then*

$$A_{\langle C_i\rangle} = \langle A_{C_i}\rangle, \qquad \textit{for all } i \geq 0.$$

Proof: Since $A_{C_i} \subseteq \forall_{\langle C_i\rangle}$, the latter obviously being a subalgebra of A, we have $\langle A_{C_i}\rangle \subseteq A_{\langle C_i\rangle}$, for all $i \geq 0$. To prove the reverse inclusion, by Lemma 1.1 we must show that for all n,

$$(*) \qquad \pi_h(A) \subseteq \langle A_{C_n}\rangle, \qquad \text{for all } h \in \langle C_n\rangle.$$

Let $h \in \langle C_n\rangle$ and denote by $l(h)$ the minimal m so that $h = k_1 \cdots k_m$, $k_i \in C_n$. We induce on $l(h)$. If $l(h) = 1$, then $h \in C_n$, hence $\pi_h(A) \subset A_{C_n} \subset \langle A_{C_n}\rangle$, and we are done. So assume $(*)$ is true for all $h \in C_n$, with $l(h) \leq m-1$. Let $l(h) = m$ and write $h = kh'$, where $h' = k_2 \cdots k_m$ and $k \in C_i$, for some $1 \leq i \leq n$. Note that if $k \in C_0$ then $kk_2 \in C_n$ thus $l(h) = m - 1$, so we are done by the induction assumption. We give a recursive formula for $\pi_{kh'}(a)$, where $k \in C_i - C_{i-1}$ some $1 \leq i \leq n$. Since $C = \bigcup C_i$ is pointed (Lemma 0.3) it follows that S is bijective on C and from [TW] that $k = y + \sum x_\alpha$, $y \in C_{i-1}$, $x_\alpha \in K_{i,\sigma,\tau}$ some $\sigma, \tau \in G$. Thus for all $a \in A$

$$(**) \qquad \pi_{kh'}(a) = \pi_{yh'}(a) + \sum \pi_{x_\alpha h'}(a).$$

Set $x = x_\alpha$; without loss of generality we may assume that $\sigma = 1$ for $xh' = (x\sigma^{-1})(\sigma h')$ and $l(\sigma h') = m - 1$. Thus $\Delta(x) = x \otimes 1 + \tau \otimes x + \sum_j x_j \otimes y_j$, $x_j, y_j \in C_{i-1}$. This implies that

$$\Delta(S^{-1}x) = 1 \otimes S^{-1}x + S^{-1}x \otimes \tau^{-1} + \sum S^{-1}y_j \otimes S^{-1}x_j,$$

and hence, since $\rho\varphi = (\varphi \otimes \mathrm{id})\Delta$, that

$$\rho\varphi(S^{-1}x) = 1 \otimes S^{-1}x + \varphi(S^{-1}x) \otimes \tau^{-1} + \sum_j \varphi(S^{-1}y_j) \otimes S^{-1}x_j.$$

Thus, for each $a \in A$,

$$\begin{aligned}\pi_{h'}(\varphi(S^{-1}x)a) &= \sum (\varphi(S^{-1}x))_0 a_0 \varphi(S((\varphi(S^{-1}x))_1 a_1)h') \\ &= \sum a_0 \varphi((Sa_1)xh') + \sum \varphi(S^{-1}x) a_0 \varphi((Sa_1)\tau h') \\ &\quad + \sum \varphi(S^{-1}y_j) a_0 \varphi((Sa_1)x_j h') \\ &= \pi_{xh'}(a) + \varphi(S^{-1}x)\pi_{\tau h'}(a) + \sum_j \varphi(S^{-1}y_j)\pi_{x_j h'}(a)\end{aligned}$$

Hence $\pi_{xh'}(a) = \pi_{h'}(\varphi(S^{-1}x)a) - \varphi(S^{-1}x)\pi_{\tau h'}(a) - \sum \varphi(S^{-1}y_j)\pi_{x_j h'}(a)$. By Lemma 1.1.2 this implies that

$$\pi_{xh'}(a) \in \pi_{h'}(A) + \pi_{S^{-1}x}(A) + \pi_{\tau h'}(A) + \pi_{S^{-1}y_j}(A)\pi_{x_j h'}(A).$$

Note that $h', S^{-1}x$ and $\tau h'$ all have length $\leq m - 1$.

Thus for $k = y + \sum x_\alpha$, $y \in C_{i-1}$ we can write $\pi_{kh'}(a)$ as a sum and product of elements of the form $\pi_{zg}(b)$, where $b \in A$, $l(g) \leq m - 1$ and $z \in C_{i-1}$. Continuing this procedure until $z \in C_0$, yields a presentation of $\pi_{kh'}(a)$ as a sum and product of elements of the form $\pi_g(b)$, where $l(g) \leq m - 1$, which by the induction assumption belongs to $\langle A_{C_n} \rangle$.

The following can be deduced from the work of H.J. Schneider [Sc] who uses the method of injective comodules and cotensor products. This point of view will be illustrated in 1.4.b which was suggested to us by the referee. We wish to thank both Schneider and the referee for their important comments.

We give explicit proofs using Lemma 1. More will be said about 1.3.2 in 3.12 for finite-dimensional H.

THEOREM 1.3: *Let H be a Hopf algebra over k, $G = G(H)$ and $C_n = \wedge^n kG$. Let A be a comodule algebra and assume there exists a total integral $\varphi\colon H \to A$.*

(1) *If $A/A^{\mathrm{co}H}$ is H-Galois and H has a bijective antipode then A_G is strongly graded (i.e. $A_G/A^{\mathrm{co}H}$ is kG-Galois).*

(2) *If A_G is strongly graded then $\langle A_{C_n}\rangle/A^{\mathrm{co}H} = A_{\langle C_n\rangle}/A^{\mathrm{co}H}$ is $\langle C_n\rangle$-Galois for all $n \geq 0$.*

Proof: (1) Let $\sigma \in G(H)$ we must show that $1 \in A_\sigma A_{\sigma^{-1}}$. Since $A/A^{\mathrm{co}H}$ is H-Galois there exist $a_i, b_i \in A$ so that

$$\sum a_{i0}b_i \otimes a_{i1} = 1 \otimes \sigma.$$

Hence $\sum \sigma S^{-1}(a_{i1}) \otimes a_{i0}b_i = 1 \otimes 1$. Thus $\sum \varphi(\sigma S^{-1}a_{i1})a_{i0}b_i = 1$. That is $\sum \pi'_\sigma(a_i)b_i = 1$, where $c_i = \pi'_\sigma(a_i) \in A_\sigma$.

Hence

$$\begin{aligned} 1 = \pi_1(1) &= \pi_1(\sum c_ib_i) \\ &= \sum c_ib_{i_0}\varphi S(\sigma b_{i_1}),\ (\text{ since } c_i \in A_\sigma) \\ &= \sum c_ib_{i_0}\varphi(S(b_{i_1})\sigma^{-1}) \\ &= \sum c_i\pi_{\sigma^{-1}}(b_i) \in A_\sigma A_{\sigma^{-1}}. \end{aligned}$$

(2) Assume $A_G/A^{\mathrm{co}H}$ is kG-Galois. We prove by induction on n that the map $\beta\colon A_{\langle C_n\rangle} \otimes_{A^{\mathrm{co}H}} A_{\langle C_n\rangle} \to A_{\langle C_n\rangle} \otimes \langle C_n\rangle$, given by $\beta(a \otimes b) = (a \otimes 1)\rho(b)$, is surjective. This is true for $n = 0$ by hypothesis, so assume it is true for $n-1$. We want to show that for all $k \in \langle C_n\rangle$, $1 \otimes k$ is in $\operatorname{Im}\beta$. If $l(k) = 1$, i.e. $k \in C_n$, then by [TW] write $k = y + \sum x_{\alpha,\sigma,\tau}$ where $y \in C_{n-1}$, $x_{\alpha,\sigma,\tau} \in K_{n,\sigma,\tau}$ some $\sigma, \tau \in G$. Now, $1 \otimes y \in \operatorname{Im}\beta$ by the inductive assumption, thus put $x = x_\alpha$. We want to show that $1 \otimes x \in \operatorname{Im}\beta$.

Let $\Delta(x) = \sigma \otimes x + x \otimes \tau + \sum x_j \otimes y_j$, $x_j, y_j \in C_{n-1}$; then $\Delta(\sigma^{-1}x) = 1 \otimes \sigma^{-1}x + \sigma^{-1}x \otimes \sigma^{-1}\tau + \sum x'_j \otimes y'_j$ $x'_j, y'_j \in C_{n-1}$ hence

$$\begin{aligned} \rho \circ \varphi(\sigma^{-1}x) &= (\varphi \otimes \mathrm{id}) \circ \Delta(\sigma^{-1}x) \\ &= 1 \otimes \sigma^{-1}x + \varphi(\sigma^{-1}x) \otimes \sigma^{-1}\tau + \sum \varphi(x'_j) \otimes y'_j. \end{aligned}$$

Thus

$$1 \otimes \sigma^{-1}x = \rho(\varphi(\sigma^{-1}x) - \varphi(\sigma^{-1}x) \otimes \sigma^{-1}\tau - \sum \varphi(x'_j) \otimes y'_j$$

and

$$1 \otimes x = (1 \otimes \sigma)\rho(\varphi(\sigma^{-1}x) - \varphi(\sigma^{-1}x) \otimes \tau - \sum \varphi(x'_j) \otimes \sigma y'_j.$$

Since $1 \otimes \sigma \in \operatorname{Im} \beta$ by hypothesis, we get that the first summand is of the form $\sum (a_i \otimes 1)\rho(b_i)\rho(\varphi(\sigma^{-1}x)) = \sum (a_i \otimes 1)\rho(b_i \varphi(\sigma^{-1}x))$, some $a_i, b_i \in A_G$. This in turn belongs to $\operatorname{Im} \beta$ since $b_i \varphi(\sigma^{-1}x) \in A_{\langle C_n \rangle}$ by 1.1. The second summand is of the form $(\varphi(\sigma^{-1}x) \otimes 1)(1 \otimes \tau)$ and $1 \otimes \tau \in \operatorname{Im} \beta$, hence the product is in $\operatorname{Im} \beta$.

The third summand is in $\operatorname{Im} \beta$ by the inductive assumption. Thus we have proved the claim for $k \in C_n$. Now induce on $l(k)$. Assume it is true for h' with $l(h') \leq m - 1$, and write $h = kh', k \in C_n$, $h' \in \langle C_n \rangle$, $l(h') = m - 1$. Then $1 \otimes h' = \sum (a_i \otimes 1)\rho(b_i)$, some $a_i, b_i \in A_{\langle C_n \rangle}$; thus

$$\begin{aligned} 1 \otimes h = (1 \otimes k)(1 \otimes h') &= \sum (a_i \otimes k)\rho(b_i) \\ &= \sum (a_i \otimes 1)(1 \otimes k)\rho(b_i). \end{aligned}$$

By the previous step, $1 \otimes k = \sum (c_j \otimes 1)\rho(d_j)$, hence the above equals $\sum (a_i c_j \otimes 1)\rho(d_j)\rho(b_i) \in \operatorname{Im} \beta$, and we are done.

Finally, β is injective since its restriction to the socle is injective. To see this, note that $\operatorname{socle}(A_{\langle C_n \rangle} \otimes_{A^{\mathrm{co}H}} A_{\langle C_n \rangle}) = A_{\langle C_n \rangle} \otimes_{A^{\mathrm{co}H}} A_G = A_{\langle C_n \rangle} \otimes_{A_G} A_G \otimes_{A^{\mathrm{co}H}} A_G$, and the restriction of β to the latter is invertible by assumption.

We end this section by specializing to certain Hopf algebras. Recall that the coradical of a Hopf algebra is the sum of its simple subcoaglebra.

THEOREM 1.4: *Let H, G, A and φ be as in 1.3, then:*

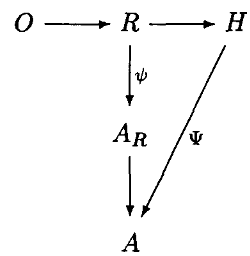
(1) *If H is a pointed Hopf algebra then $A/A^{\mathrm{co}H}$ is H-Galois if and only if A_G is strongly graded by G.*

(2) *If H is a Hopf algebra such that the coradical R is a Hopf subalgebra of H then $A/A^{\mathrm{co}H}$ is H-cleft if and only if $A_R/A^{\mathrm{co}H}$ is R-cleft.*

Proof: (1) follows from Theorem 1.3 and Lemma 0.3.

(2) can be deduced Theorem 1.3, [Sc. 2.1] and [DT]. Directly, using the following proof of the referee, suppose $A_R/A^{\mathrm{co}H}$ is R-cleft. Thus there exists an invertible R-comodule map $\psi: R \to A_R$.

Since A is an injective H-comodule the following diagram can be completed by an H-comodule map $\Psi: H \to A$:



By [T] this Ψ is invertible (under convolution), since $\Psi|_R = \psi$ is invertible. Hence $A/A^{\mathrm{co}H}$ is H-cleft. The reverse direction is easy.

More about actions of pointed Hopf algbras will be proved in the next sections.

2. Semiinvariants of actions

In this section assume H is a finite-dimensional Hopf algebra. If M is a left H-module then it is a right H^*-comodule, so section 1 applies, though in certain cases we give a more illuminating module theoretic description. First, some connections between the module and comodule structures via the left integrals t, T of H and H^* respectively. Let $\lambda \in G(H^*)$ and $\Lambda \in G(H)$ be the distinguished group-likes, that is $th = t\langle\lambda, h\rangle$, all $h \in H$, and $Tx = T\langle\Lambda, x\rangle$ all $x \in H^*$.

Lemma 2.1: *Let M be a left H-module and A an H-module algebra, then:*

(1) *For all $h \in H$, $x \in H^*$*

$$h(x \rightharpoonup t) = \sum \langle S^{-1}x_1, h\rangle x_2 \rightharpoonup t,$$

and

$$(x \rightharpoonup t)h = \sum \langle \lambda S^{-1}x_2, h\rangle x_1 \rightharpoonup t;$$

(2) *if $x \in H^*$ and $m \in M$ then*

$$\rho((x \rightharpoonup t)\cdot m) = \sum (x_2 \rightharpoonup t)\cdot m \otimes S^{-1}x_1;$$

(3) *if A contains an element c of trace 1, and $\rho(a) = \sum a_i \otimes x_i$, then*

$$a = \sum (Sx_i \rightharpoonup t)\cdot(ca_i).$$

Proof:

(1)
$$\begin{aligned} h(x \rightharpoonup t) &= \sum x_2 \rightharpoonup ((S^{-1}x_1 \rightharpoonup h)t) \quad \text{(by 0.11)} \\ &= \sum x_2 \rightharpoonup \varepsilon(S^{-1}x_1 \rightharpoonup h)t \\ &= \sum x_2 \rightharpoonup \langle S^{-1}x_1, h\rangle t. \end{aligned}$$

The other formula is proved similarly via 0.10:

$$\tag{2}\begin{aligned}\rho((x\rightharpoonup t)\cdot m) &= \sum h_i(x\rightharpoonup t)\cdot m\otimes h_i^* \\ &= \sum(x_2\rightharpoonup t)\cdot m\otimes\sum_i\langle S^{-1}x_1,h_i\rangle h_i^* \quad \text{(by (1))} \\ &= \sum(x_2\rightharpoonup t)\cdot m\otimes S^{-1}x_1.\end{aligned}$$

$$\tag{3}\begin{aligned}a = (t\cdot c)a &= \sum t_1\cdot(cSt_2\cdot a) \quad \text{(by 0.10)} \\ &= \sum t_1\cdot(c\sum_i\langle x_i,St_2\rangle a_i) \\ &= \sum_i\langle x_i,St_2\rangle t_1\cdot(ca_i) \\ &= \sum_i(Sx_i\rightharpoonup t)\cdot(ca_i).\end{aligned}$$

Recall the notation: $M_K=\rho^{-1}(M\otimes K)$.

COROLLARY 2.2: *Let M be a left H-module, A a left H-module algebra and K a subcoalgebra of H^*; then:*

(1) *$H_K=S(K)\rightharpoonup t$ (where H is a left H-module by multiplication) and H_K is an ideal of H.*

(2) *$H_K\cdot M\subset M_K$.*

(3) *If A contains an element of trace 1, then $H_K\cdot A=A_K=\sum_{k\in K}\pi_k(A)$.*

Proof: (1) By 2.1.(2) with $m=1$, we get for $x\in K$,

$$\rho(S(x)\rightharpoonup t)=\sum(S(x_1)\rightharpoonup t)\otimes x_2\in H\otimes K.$$

Thus $S(K)\rightharpoonup t\subset\rho^{-1}(H\otimes K)=H_K$. Conversely, let $h\in H_K$, and let $p\in H^*$ so that $S(p)\rightharpoonup t=h$. Then as above,

$$\rho(h)=\rho(S(p)\rightharpoonup t)=\sum(S(p_1)\rightharpoonup t)\otimes p_2\in H\otimes K,$$

hence $p\in K$ and so $h\in S(K)\rightharpoonup t$. The fact that H_K is an ideal of H follows from 2.1(1).

(2) follows from 2.1(2).

(3) follows from 2.1(3), and from 1.1.

Specializing to $K=k\sigma$, and denoting $\sigma\rightharpoonup t=t^\sigma$, we have:

COROLLARY 2.3: *Let M, A be as in 2.2 and $\sigma \in G(H^*)$ then:*

(1) $kt^{\sigma^{-1}} = \rho^{-1}(H \otimes \sigma) = \{x \in H |\, hx = \langle\sigma, h\rangle x, \quad \text{all } h \in H\} \equiv H_\sigma$, *an ideal of H.*

(2) $t^{\sigma^{-1}} \cdot M \subset \rho^{-1}(M \otimes \sigma) = \{m \in M |\, h \cdot m = \langle\sigma, h\rangle m, \quad \text{all } h \in H\} \equiv M_\sigma$.

(3) *If A contains an element of trace 1 then $t^{\sigma^{-1}} \cdot A = A_\sigma$.*

Specializing to $K = k\tau \wedge k\sigma$ we have analogously:

PROPOSITION 2.4: *Let M, A be as in 2.2, and $\sigma, \tau \in G(H^*)$. Denote $M_{\sigma,\tau} = M_{k\tau\wedge k\sigma}$, then:*

(1) $M_{\sigma,\tau} = \{a \in M |\, \rho(a) = a \otimes \sigma + \sum b_i \otimes x_i, \text{where}\, b_i \in M_\tau,\ x_i \in P_{\sigma,\tau}(H^*)\}$.

(2) $M_{\sigma,\tau} = \{a \in M |\, rq \cdot a = \langle\tau, r\rangle q \cdot a + \langle\sigma, q\rangle r \cdot a - \langle\tau, r\rangle\langle\sigma, q\rangle a \text{ all } r, q \in H\}$.

(3) $((k\sigma^{-1} \wedge k\tau^{-1}) \rightharpoonup t) \cdot M = H_{\sigma,\tau} \cdot M \subset M_{\sigma,\tau}$.

(4) *If A contains an element of trace 1, then $H_{\sigma,\tau} \cdot A = A_{\sigma,\tau} = \sum_{h \in k\tau\wedge k\sigma} \pi_h(A)$.*

(5) $H_{\sigma,\tau} = (k\sigma^{-1} \wedge k\tau^{-1}) \rightharpoonup t$ *is an ideal of H,*

$$H_{\sigma,\tau} \rightharpoonup T = H_{\sigma,\tau} \rightharpoonup H^* = k\tau \wedge k\sigma$$

and $(H_{\sigma,\tau})^\mu = H_{\sigma\mu^{-1},\tau\mu^{-1}}$, *for all $\mu \in G(H^*)$.*

(6) $S(H_{\sigma,\tau}) = H_{(\lambda\tau)^{-1},(\lambda\sigma)^{-1}}$, *where λ is the group-like associated with t.*

Proof: (1) Since $k\tau \wedge k\sigma = k\sigma \oplus P_{\sigma,\tau}(H^*)$ we must show that if

$$\rho(a) = a' \otimes \sigma + \sum b_i \otimes x_i,$$

where $\{x_i\}$ are linearly independent elements of $P_{\sigma,\tau}(H^*)$, then $a = a'$ and $b_i \in M_\tau$. First, since $a = (\mathrm{id} \otimes \varepsilon)\rho(a)$, and since $\varepsilon(x_i) = 0$ we get $a = a'$. Next we show that $\rho(b_i) = b_i \otimes \tau$, and hence $b_i \in M_\tau$. Well, $(\mathrm{id} \otimes \Delta)\rho(a) = a \otimes \sigma \otimes \sigma + \sum b_i \otimes [x_i \otimes \sigma + \tau \otimes x_i]$ while

$$\begin{aligned}(\rho \otimes \mathrm{id})\rho(a) &= \rho(a) \otimes \sigma + \sum \rho(b_i) \otimes x_i \\ &= (a \otimes \sigma \otimes \sigma + \sum b_i \otimes x_i \otimes \sigma) + \sum \rho(b_i) \otimes x_i.\end{aligned}$$

Since $(\mathrm{id} \otimes \Delta)\rho(a) = (\rho \otimes \mathrm{id})\rho(a)$, we get $\sum b_i \otimes \tau \otimes x_i = \sum \rho(b_i) \otimes x_i$ and hence that $\rho(b_i) = b_i \otimes \tau$ for each i.

(2) Let a belong to the right hand side and let $\{h_i\}$, $\{h_i^*\}$ be dual bases of H and H^* respectively, then

$$\begin{aligned}(id\otimes\Delta)\rho(a) =&(\rho\otimes \mathrm{id})(\rho(a)) = \sum_{ij} h_ih_j\cdot a\otimes h_i^*\otimes h_j^* \\ =&\sum\langle\tau,h_i\rangle h_j\cdot a\otimes h_i^*\otimes h_j^* \\ &+\sum\langle\sigma,h_j\rangle h_i\cdot a\otimes h_i^*\otimes h_j^* \\ &-\sum_{i,j}\langle\tau,h_j\rangle\langle\sigma,h_i\rangle a\otimes h_i^*\otimes h_j^* \\ =&\sum h_j\cdot a\otimes\tau\otimes h_j^* \\ &+\sum h_i\cdot a\otimes h_i^*\otimes\sigma - a\otimes\sigma\otimes\tau \\ =&\sum h_i\cdot a\otimes[h_i^*\otimes\sigma+\tau\otimes h_i^*] - a\otimes\sigma\otimes\tau.\end{aligned}$$

Hence $\rho(a)\subset M\otimes(k\tau\wedge k\sigma)$, and so $a\in M_{\sigma,\tau}$.

The reverse inclusion is straightforward using (1) and the connection between ρ and the action.

(3) and (4) follow directly from 2.1.

(5)
$$\begin{aligned}H_{\sigma,\tau} &= S(k\tau\wedge k\sigma)\rightharpoonup t \text{ (by 2.2)} \\ &= (k\sigma^{-1}\wedge k\tau^{-1})\rightharpoonup t.\end{aligned}$$

$H_{\sigma,\tau}$ is an ideal of H by 2.2. Now, since H^* is an H-module algebra with an element of trace 1, 2.2 applies with $\rho=\Delta_{H^*}$, and thus $H_{\sigma,\tau}\rightharpoonup H^* = H^*_{k\tau\wedge k\sigma}$. But $k\tau\wedge k\sigma$ is a subcoalgebra of H^*, hence

$$k\tau\wedge k\sigma = \rho^{-1}(H^*\otimes(k\tau\wedge k\sigma)) = H^*_{k\tau\wedge k\sigma}.$$

We have shown that $H_{\sigma,\tau}\rightharpoonup H^* = k\tau\wedge k\sigma$.

Since $H^*=H\rightharpoonup T$, and since $H_{\sigma,\tau}$ is an ideal of H, we have also

$$H_{\sigma,\tau}\rightharpoonup T = k\tau\wedge k\sigma.$$

Finally, since $H_{\sigma,\tau}=(k\sigma^{-1}\wedge k\tau^{-1})\rightharpoonup t$ we have

$$\begin{aligned}(H_{\sigma,\tau})^\mu &= \mu(k\sigma^{-1}\wedge k\tau^{-1})\rightharpoonup t \\ &= (k\mu\sigma^{-1}\wedge k\mu\tau^{-1})\rightharpoonup t \\ &= H_{\sigma\mu^{-1},\tau\mu^{-1}},\end{aligned}$$

$$(6)\qquad \begin{aligned}(H_{\tau^{-1},\sigma^{-1}} \rightharpoonup T) \rightharpoonup t &= (k\sigma^{-1} \wedge k\tau^{-1}) \rightharpoonup t \quad \text{(by 5)}\\ &= \rho^{-1}(H \otimes (k\tau \wedge k\sigma))\\ &= H_{\sigma,\tau}.\end{aligned}$$

On the other hand, by [CF, 0.5]

$$(H_{\tau^{-1},\sigma^{-1}} \rightharpoonup T) \rightharpoonup t = S^{-1}((H_{\tau^{-1},\sigma^{-1}})^{\lambda}).$$

Thus

$$\begin{aligned}S(H_{\sigma,\tau}) &= (H_{\tau^{-1},\sigma^{-1}})^{\lambda}\\ &= H_{\tau^{-1}\lambda^{-1},\sigma^{-1}\lambda^{-1}} = H_{(\lambda\tau)^{-1},(\lambda\sigma)^{-1}} \quad \text{(by (5))}.\end{aligned}$$

LEMMA 2.5: *Let H be a finite-dimensional Hopf algebra, and $\sigma \in G$. Let A be an H-module algebra. Then:*

1. *If $\sigma \neq \varepsilon$ then $\langle \sigma, t\rangle = 0$ and so if $\tau \in G$ and $\tau \neq \sigma$ then $t^\tau t^\sigma = 0$.*
2. *If $x \in A$, $y \in A_\sigma$ then for all $h \in H$, $h \cdot (xy) = (h^\sigma \cdot x)y$ (thus Ay is an H-stable left ideal of A).*
3. *If $x \in A_\sigma$ is invertible in A then $x^{-1} \in A_{\sigma^{-1}}$.*
4. *$t^\sigma \cdot A_\tau = 0$ if $\sigma \neq \tau^{-1}$.*
5. *If A contains an element c of trace 1, then for all $a \in A$, $t^\sigma \cdot A = A_{\sigma^{-1}}$, moreover $\pi_\sigma(a) = \langle \Lambda, \sigma\rangle t^{\sigma^{-1}} \cdot (ac)$ (where Λ is the distinguished group-like associated with T).*

Proof: (1) For all $h \in H$, $\langle \sigma, ht\rangle = \langle \sigma, h\rangle\langle \sigma, t\rangle$. However, $\langle \sigma, ht\rangle = \varepsilon(h)\langle \sigma, t\rangle$. Thus if $\langle \sigma, t\rangle \neq 0$, then $\langle \sigma, h\rangle = \varepsilon(h)$, all $h \in H$. That is $\sigma = \varepsilon$. Thus $\langle \sigma, t\rangle = 0$ as claimed. Now if $\tau \neq \sigma$ then $t^\tau t^\sigma = \langle \sigma^{-1}, t^\tau\rangle t^\sigma$. But $\langle \sigma^{-1}, t^\tau\rangle = \langle \sigma^{-1}\tau, t\rangle = 0$, since $\sigma^{-1}\tau \neq \varepsilon$.

(2) Let $h \in H$, $x \in A$, $y \in A_\sigma$, then

$$\begin{aligned}h \cdot (xy) &= \sum (h_1 \cdot x)(h_2 \cdot y)\\ &= \sum (h_1 \cdot x)\langle \sigma, h_2\rangle y = (h^\sigma \cdot x)y.\end{aligned}$$

(3) Let $x \in A_\sigma$ be invertible in A, and let $h \in H$. Then

$$\begin{aligned}\langle \sigma^{-1}, h\rangle &= \varepsilon(h^{\sigma^{-1}}) = h^{\sigma^{-1}} \cdot 1 = h^{\sigma^{-1}} \cdot (x^{-1}x)\\ &= ((h^{\sigma^{-1}})^\sigma \cdot x^{-1})x \quad \text{(by (2))}.\end{aligned}$$

Hence $\langle \sigma^{-1}, h\rangle = (h \cdot x^{-1})x$, which implies that $h \cdot x^{-1} = \langle \sigma^{-1}, h\rangle x^{-1}$. That is $x^{-1} \in A_{\sigma^{-1}}$.

(4) Let $x \in A_\tau$, then $t^\sigma \cdot x = \langle \tau, t^\sigma \rangle x = \langle \sigma\tau, t\rangle x$. So if $\sigma\tau \neq \varepsilon$, $\langle \sigma\tau, t\rangle = 0$ (by (1)) and hence $t^\sigma \cdot x = 0$.

(5) By [CF] if $\sigma \in H$, then $\rho(a) = \sum t_1 \cdot a \otimes S^{-1}(t_2 \rightharpoonup T)$ and φ: $H^* \to A$ is defined by $\varphi(T) = c$, hence

$$\begin{aligned}\pi_\sigma(a) &= \sum (t_1 \cdot a)\varphi(SS^{-1}(t_2 \rightharpoonup T)\sigma)\\ &= \sum (t_1 \cdot a)\varphi((t_2 \rightharpoonup T)\sigma)\\ &= \sum (t_1 \cdot a)\varphi(\langle \Lambda S^{-1}t_3, \sigma\rangle t_2 \rightharpoonup T) \quad \text{(by 2.1)}\\ &= \sum (t_1 \cdot a)\varphi(t_2 \rightharpoonup T\langle S^{-1}t_3, \sigma\rangle\langle \Lambda, \sigma\rangle\\ &= \langle \Lambda, \sigma\rangle \sum (t_1 \cdot a)(t_2 \cdot \varphi(T))\langle t_3, \sigma^{-1}\rangle\\ &= \langle \Lambda, \sigma\rangle t^{\sigma^{-1}} \cdot (ac).\end{aligned}$$

More can be said if H is semisimple.

LEMMA 2.6: *Let H be a finite-dimensional semisimple Hopf algebra. Then:*

(1) *$\{t^\sigma\}_{\sigma \in G}$ is a set of orthogonal idempotents.*

(2) *For each $\sigma \in G$, $t^\sigma \cdot A = A_{\sigma^{-1}}$.*

Proof: (1) By 2.1 the set is orthogonal. So it remains to be seen that t^σ is an idempotent. This is immediate from the fact that t is an idempotent and $t \rightharpoonup t^\sigma$ is an automorphism.

(2) Follows from 2.3 since $1 \in A$ is an element of trace 1.

The following deals with the situation of Theorem 1.3 for finite-dimensional H. Some more can be said here.

THEOREM 2.7: *Let H be a finite-dimensional Hopf algebra over a field k, acting on A so that A/A^H is an H^*-Galois extension; then:*

(i) *$A_\sigma \neq 0$ for all $\sigma \in G = G(H^*)$.*

If moreover A has an element of trace 1, then:

(ii) *A_G is strongly graded.*

(iii) *The algebras A^H, $A\#H$ and $A_G\#(kG)^*$ are Morita equivalent.*

(iv) *If H is semisimple then $A_G\#(kG)^*$ is isomorphic as an algebra to an algebra $S \subset A\#H$ where $1_S = \sum_{\sigma\in G} t^\sigma$ (hence S is not necessarily a subalgebra of $A\#H$).*

Proof: (i) Since A/A^H is H^*-Galois, let $\{x_i, y_i\} \subset A$ be such that $\sum x_i t y_i = 1$ in $A\#H$. Thus we have $\sum x_i t^\sigma y_i = (\sum x_i t y_i)^\sigma = 1^\sigma = 1$, Applying this to 1_A

we get:

$$(*) \qquad \sum x_i t^\sigma \cdot y_i = 1, \quad \text{for each } \sigma \in G$$

Thus, $t^\sigma \cdot A \neq 0$ for each $\sigma \in G$. However, by 2.3, $t^\sigma \cdot A \subset A_{\sigma^{-1}}$, and hence $A_\sigma \neq 0$ for all $\sigma \in G$.

(ii) follows from 1.3.

(iii) By (ii) and [CM] $A_G\#(kG)^*$ is Morita equivalent to $A_1 = A^H$. By hypothesis $A\#H$ is Morita equivalent to A^H as well, hence $A\#H$ is Morita equivalent to $A_G\#(kG)^*$.

(iv) Let H be semisimple; then by 2.4 $\{t^\sigma\}_{\sigma\in G} \subset H$ is a set of orthogonal idempotents. Denote by a_τ an element of A_τ. Then

$$t^\sigma \cdot a_\tau = \begin{cases} a_\tau & \text{if } \tau = \sigma^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{p_\sigma\}_{\sigma\in G}$ denote the basis of $(kG)^*$ dual to the basis $\{\sigma\}_{\sigma\in G}$ of kG. Let $f\colon A_G\#(kG)^* \to A\#H$ be defined by

$$f\colon a_\tau\#p_\sigma \to a_\tau t^{\sigma^{-1}}, \qquad \text{all } \sigma,\tau \in G, \quad \text{all } a_\tau \in A_\tau.$$

We claim that f is multiplicative. To see this recall [CM] that

$$(a_\tau\#p_\sigma)(a_\mu\#p_\lambda) = \begin{cases} a_\tau a_\mu\#p_\lambda, & \text{if } \mu = \sigma\lambda^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$f((a_\tau\#p_\sigma)(a_\mu\#p_\lambda)) = \begin{cases} a_\tau a_\mu t^{\lambda^{-1}}, & \text{if } \mu = \sigma\lambda^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

It is now straightforward to check that this equals $f(a_\tau\#p_\sigma)f(a_\mu\#p_\lambda)$.

By freeness of $A\#H$ over A and 2.6(1), f is injective. So $A_G\#(kG)^* \cong_{\text{alg}} \operatorname{Im} f \equiv S$. Note that $1_S = \sum_{\sigma\in G} t^\sigma$ which may very well be different from 1_A.

Let us apply the above to a situation described in [CFM, BeCF].

THEOREM 2.8: *Let H be a finite-dimensional Hopf algebra acting on A. Let $G = G(H^*)$ and $\Phi = \{\sigma \in G\colon A_\sigma \neq 0\}$. If A has no H-stable left ideals (i.e. A is an irreducible $A\#H$-module), then:*

(i) *Φ is a subgroup of G.*

(ii) *$A_G = A^H\#_\mu k\Phi$ (where μ is an invertible normal cocycle) and A^H is a division ring.*

(iii) *If moreover the right dimension of A over $A^H = \dim_k H$, then $\Phi = G$ and so $A_G = A^H\#_\mu kG$.*

(iv) *If H is pointed and $\Phi = G$ then $A = A^H\#_\gamma H$.*

Remark: If $A = D$ is a division ring then it is an irreducible $D\#H$-module, hence by Theorem 2.8, $D_G = D^H\#_\mu k\Phi$. If $[D : D^H] = \dim_k H$ then $D_G = D^H\#_\mu kG$.

Proof: (i) Let $\sigma \in \Phi$, then for each $0 \neq x_\sigma \in A_\sigma$, Ax_σ is by Lemma 2.5 an H-stable left ideal of A, hence equals A. Thus each $0 \neq x_\sigma \in A_\sigma$ is invertible in A and so by Lemma 2.5 $(x_\sigma)^{-1} \in A_{\sigma^{-1}}$. Let $\sigma, \tau \in \Phi$ then $0 \neq x_\tau(x_\sigma)^{-1} \in A_{\tau\sigma^{-1}}$, implying that $\tau\sigma^{-1} \in \Phi$. Thus Φ is a subgroup of G and A_G is graded by Φ so that each A_σ contains an invertible element of A_G.

(ii) A^H is a division ring by [BeCF]. Since each A_σ contains an invertible element of A_G, [*Da*] implies that A_G is a crossed product, $A_G = A^H\#_\mu k\Phi$, some normal invertible cocycle μ.

(iii) Finally, if $[A : A^H] = \dim_k H$, then by [BCM 3.3], A/A^H is H^*-Galois hence $\Phi = G$ by 2.7.

(iv) By (ii) A_G is kG-cleft, hence by 1.4 A is H-cleft.

3. Module categories

If R is a ring, then R-module will mean a left-R module, denoted by ${}_RM$ and the category of R-modules will be denoted by $R -$ Mod. In this section we consider connections between A^H, A, A_G and $A\#H -$ Mod. First, we show that as in the previous sections, pointed Hopf algebras lead to a particularly nice such connection. Starting with $H = kG$, G a finite group, it is a fact that $A \subset A\#H$ is a finite normalizing extension that leads to a strong connection between A and $A\#H -$ Mod [LP]. On the other hand, if L is a restricted Lie algebra acting as derivations on A, $\mathrm{char}(k) = p \neq 0$, then $A\#u(L)$ ($u(L) =$ the restricted enveloping algebra) is a finite subnormalizing extension of A [W, 1.1.3], which is sufficient to give strong connections between A and $A\#H -$ Mod [Q].

Definition 3.1: Let R be a subring of a ring S, then $R \subset S$ is said to be a **finite subnormalizing extension** (or triangular extension [L]) if there exists $x_i \in S$ so that $S = \sum_{i=1}^n Rx_i = \sum_{i=1}^n x_iR$, and for each $1 \leq j \leq n$, $\sum_{i=1}^j Rx_i = \sum_{i=1}^j x_iR$. ∎

In the following proposition we use coradical filtrations as in the previous section. We wish to thank D. Quinn for his suggestion concerning the above.

THEOREM 3.2: *Let H be a finite-dimensional pointed Hopf algbra acting on A. Then $A\#H$ is a finite subnormalizing extension of A.*

Proof: We will construct by induction a k-basis of H, $\{x_1, \ldots, x_n\}$ such that $\sum_{i=1}^{j} x_i A = \sum_{i=1}^{j} A x_i$, $1 \le j \le n$. Let $G = G(H)$, and $kG = C_0 \subset \cdots \subset C_m = H$ be the coradical filtration of H.

First, G forms a k-basis for kG, which clearly normalizes A in $A\#H$. So let $\{x_1, \ldots, x_t\} = G$ and assume that a basis $\{x_1, \ldots, x_r\}$ of C_{i-1} has been constructed including $\{x_1, \ldots, x_t\}$ such that $\sum_{i=1}^{j} x_i A = \sum_{i=1}^{j} A x_i$, $j = 1, \ldots, r$. Since $C_i = C_{i-1} \oplus \sum K_{i,\sigma,\tau}$ let $x \in K_{i,\sigma,\tau}$, that is $\Delta(x) = x \otimes \sigma + \tau \otimes x + \sum u_m \otimes v_m$, where $u_m, v_m \in C_{i-1}$. By (0.12), for all $a \in A$, $h \in H$ $ah = \sum h_2(S^{-1}(h_1) \cdot a)$. Thus for $h = x$, we get

$$ax = \sigma(S^{-1}(x) \cdot a) + x(\tau^{-1} \cdot a) + \sum v_m(S^{-1}(u_m) \cdot a).$$

On the other hand,

$$xa = (x \cdot a)\sigma + (\tau \cdot a)x + \sum (u_m \cdot a) v_m.$$

Thus setting $x_{r+1} = x$, we have: $Ax_{r+1} \in \sum_{n=1}^{r+1} x_n A$ and $x_{r+1}A \in \sum_{n=1}^{r+1} A x_n$. If $C_i = C_{i-1} + kx_{r+1}$, we are done. If not, take $x_{r+2} \in K_{i,\sigma,\tau} \backslash (C_{i-1} + kx_{r+1})$, some $\sigma, \tau \in G$, and continue as above. The procedure will stop since H is finite-dimensional.

Corollary 3.3: *Let H be a finite-dimensional pointed Hopf algebra acting on A. Let $M \in A\#H$ – Mod. Then:*

(i) *If ${}_{A\#H}M$ is Noetherian, then ${}_A M$ is Noetherian.*
(ii) *If ${}_{A\#H}M$ is Artinian, then ${}_A M$ is Artinian.*
(iii) *If ${}_{A\#H}M$ has a Krull dimension, then ${}_A M$ has a Krull dimension, and the dimensions are equal.*

Proof: By Theorem 3.2 and [L, theorem 1.2]. Also see [W], [L] where many other properties were proved for subnormalizing extensions.

In the following we use 3.2 and a Hopf algebra version of the technique used by [CQ].

Corollary 3.4: *Let H be a finite-dimensional Hopf algebra acting on A, and assume H^* is pointed, A/A^H is H^*-Galois, and A has an element of trace 1. Let $M \in A$ – Mod, then:*

(i) *If ${}_A M$ is Noetherian, then ${}_{A^H} M$ is Noetherian.*
(ii) *If ${}_A M$ is Artinian, then ${}_{A^H} M$ is Artinian.*

(iii) *If ${}_AM$ has a Krull dimension, then ${}_{A^H}M$ has a Krull dimension, and the dimensions are equal.*

Proof: We prove only (i); the rest follow similarly.

Since the induction functor

$$A\#H\otimes_A - : A - \mathrm{Mod} \to (A\#H)\#H^* - \mathrm{Mod}$$

is an equivalence of categories, it follows that $A\#H\otimes_A M$ is Noetherian as an $(A\#H)\#H^*$-module. By 3.3, and since H^* is pointed, $A\#H\otimes_A M$ is Noetherian as an $A\#H$-module. But we have

$$A\otimes_{A^H} M\cong A\otimes_{A^H}(A\otimes_A M)\cong(A\otimes_{A^H} A)\otimes_A M. \tag{1}$$

On the other hand, since A/A^H is H^*-Galois, then as mentioned in section 0, $[\ ,\ ]: A\otimes_{A^H} A\to A\#H$ is an isomorphism of $A\#H$-modules. Hence:

$$[\ ,\ ]\otimes 1_M: (A\otimes_{A^H} A)\otimes_A M\to A\#H\otimes_A M \tag{2}$$

is an isomorphism of $A\#H$ modules, and the composition of (1) and (2) gives an isomorphism of $A\#H$ modules $A\otimes_{A^H} M\cong A\#H\otimes_A M$, which is functorial. Thus $A\otimes_{A^H} M$ is a Noetherian $A\#H$-module. Since the induced functor

$$A\otimes_{A^H} -: A^H - \mathrm{Mod}\to A\#H - \mathrm{Mod}$$

is an equivalence, by hypothesis, we conclude that ${}_{A^H}M$ is Noetherian.

We remark here that for Hopf algebras which are not necessarily pointed, it was shown in [C1] that if the action of H on A is inner then $A\subset A\#H$ is a centralizing extension, and an analogue of 3.3 was proved using this fact.

Since cocommutative Hopf algebras over an algebraically closed field are pointed, 3.2 applies. We ask whether the condition on the field can be omitted, and more generally:

Question 3.5: Let (H,R) be a finite-dimensional quasitriangular [D] Hopf algebra over a field k, acting on A; is $A\subset A\#H$ a subnormalizing extension? ∎

Subnormalizing extensions were considered in [Q] with applications to actions of $u(L)$; analogous proofs can be applied to the following, resulting from 3.2.

COROLLARY 3.6: *Let H be a finite-dimensional pointed Hopf algebra acting on A, then:*

(1) *$A\#H$ is fully integral over A (that is, given $r_1, \ldots, r_m \in A\#H$ we get $r_1 r_2 \ldots r_m = \psi$ where ψ is a sum of A-monomials in the r_i of degree less than m).*

(2) *If Q is an H-prime ideal of A and P is a prime ideal of $A\#H$ with $P \cap A \subset Q$, then there exists a prime ideal P' of $A\#H$ with $P \subset P'$ and $P' \cap A = Q$ (i.e. there is a "going up" property).*

We end this section by considering a connection between $A_G\#(kG)^* - \mathrm{Mod}$ and $A\#H - \mathrm{Mod}$ when some of the conditions are relaxed.

We will use the following result, due to P. Gabriel.

LEMMA 3.7 ([Fa] Proposition 15.18): *If $F\colon B - \mathrm{Mod} \to A - \mathrm{Mod}$ has a right adjoint G, such that $F \circ G \cong 1_A - \mathrm{Mod}$ canonically, and F is an exact functor, then*

$$\mathrm{Ker}(F) = \{X \in B - \mathrm{Mod} \colon F(X) = 0\}$$

is a localizing subcategory of $B - \mathrm{Mod}$ and F induces an equivalence

$$B - \mathrm{Mod} \,/\, \mathrm{Ker}\, F \xrightarrow{\simeq} A - \mathrm{Mod}\,.$$

We begin with the following simple result, which might be known. Nevertheless, since we have not been able to find a reference, we include a proof and a consequence.

PROPOSITION 3.8: *Let $\langle A, B, {}_A P_B, {}_B Q_A, \alpha, \beta\rangle$ be a Morita context*

$$\alpha\colon P \otimes Q \to A \quad \alpha(p \otimes q) = [p, q],$$
$$\beta\colon Q \otimes P \to B \quad \beta(q \otimes p) = (q, p).$$

Suppose α is surjective. Then $F = P_B \otimes - : B - \mathrm{Mod} \to A - \mathrm{Mod}$ induces an equivalence $B - \mathrm{Mod} \,/\, \mathrm{Ker}(F) \xrightarrow{\simeq} A - \mathrm{Mod}$ (i.e. $A - \mathrm{Mod}$ is equivalent with a quotient of $B - \mathrm{Mod}$).

Proof: We will apply Lemma 3.7 to F. By [Fa] 11.2.7.3 P_B is projective, hence flat, so F is exact. F has a right adjoint

$$G = \mathrm{Hom}_A(P, -)\colon A - \mathrm{Mod} \to B - \mathrm{Mod}\,.$$

The canonical morphism, for all $M \in A-\mathrm{Mod}$, from $FG(M) = P \otimes_B \mathrm{Hom}_A(P, M)$ to M sends $p \otimes f$ to $f(p)$. Let us show it is an isomorphism.

Let u_i, v_i be such that $\sum[u_i, v_i] = 1$ $(u_i \in P, v_i \in Q)$,

$$\psi\colon P \otimes_B \mathrm{Hom}_A(P, M) \to M \quad \psi(p \otimes f) = f(p).$$

Define $\varphi\colon M \to P \otimes_B \mathrm{Hom}_A(P, M)$ by

$$\varphi(m) = \sum u_i \otimes f_i$$

where $m \in M$ and $f_i(p) = [p, v_i]m \ \ \forall p \in P$. Clearly all f_i's are A-linear. Now if $m \in M$, $(\psi \circ \varphi)(m) = \psi(\sum u_i \otimes f_i) = \sum f_i(u_i) = \sum[u_i, v_i]m = m$.

If $p \otimes f \in P \otimes_B \mathrm{Hom}_A(P, M)$, then

$$(\varphi \circ \psi)(p \otimes f) = \varphi(f(p)) = \sum u_i \otimes f_i$$

where

$$\begin{aligned} f_i(p') &= [p', v_i]f(p) = f([p', v_i]p) \\ &= f(p'(v_i, p)) = ((v_i, p)f)(p'), \quad \text{for all } p' \in P, \end{aligned}$$

so $f_i = (v_i, p)f$.

Thus

$$\begin{aligned} (\varphi \circ \psi)(p \otimes f) &= \sum u_i \otimes (v_i, p)f = \sum u_i(v_i, p) \otimes f \\ &= \sum[u_i, v_i]p \otimes f = p \otimes f, \end{aligned}$$

and we are done.

COROLLARY 3.9: *Let $\langle A, B, P, Q, \alpha, \beta\rangle$ be a Morita context with $\alpha\colon P \otimes Q \to A$ surjective. Then β is surjective $\Leftrightarrow P_B$ is faithfully flat.*

Proof:

β is surjective $\Leftrightarrow P_B \otimes -$ is an equivalence $\Leftrightarrow$

$\Leftrightarrow \mathrm{Ker}(P_B \otimes -) = \{X \in B-\mathrm{Mod}: P \otimes_B X = 0\} = 0 \Leftrightarrow$

$\Leftrightarrow P_B$ is faithfully flat.

We remark that Corollary 3.9 provides a new proof for [KT, 1.9, (1)$\Leftrightarrow$(5)].

COROLLARY 3.10: *If H is finite dimensional and A is an H-module algebra, then:*

(i) *If A/A^H is H^*-Galois, then $A^H - \mathrm{Mod}/\mathcal{C} \cong A\#H - \mathrm{Mod}$. for some localizing subcategory $\mathcal{C}$ of $A^H - \mathrm{Mod}$.*

(ii) *If A has an element of trace 1, then* $A^H - \mathrm{Mod} \cong A\#H - \mathrm{Mod}/\mathcal{C}'$ *for some localizing subcategory* $\mathcal{C}'$ *of* $A\#H - \mathrm{Mod}$.

Proof: Recall $\langle A^H, A\#H, A, A, \alpha, \beta\rangle$ $\alpha\colon A \otimes_{A^H} A \to AH$, $\beta\colon A \otimes_{A\#H} A \to A^H$ is a Morita context, and α is surjective $\Leftrightarrow A/A^H$ is H^*-Galois, and β is surjective $\Leftrightarrow A$ has an element of trace 1. Apply now Proposition 3.8, and we are done.

Recall that if A/A^H is H^*-Galois and A has an element of trace 1 then A^H and $A\#H$ are Morita equivalent. Corollary 3.10 shows what happens if we drop one of the conditions. We remark that the proof of (i) is a different proof for Corollary 2.3, (i) of [CRVO]. Assertion (ii) is a generalization of Corollary 2.3 (ii) of [CRVO], where H was assumed to be semisimple. The new approach makes these results completely symmetric (see also [ZVO]).

Corollary 3.2 of [CRVO] states that if H is finite dimensional and semisimple, then $A_G\#(kG)^*$ is equivalent with a quotient category of $A\#H - \mathrm{Mod}$. The following is a dual version of this result.

COROLLARY 3.11: *If A/A^H is H^*-Galois, then* $A\#H - \mathrm{Mod}$ *is equivalent with a quotient category of* $A_G\#(kG)^* - \mathrm{Mod}$.

Proof: Denote $A_G\#(kG)^* = P$ and $A_G = S$. We have the following Morita contexts:

$\langle A^H, A\#H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, \alpha, \beta\rangle$ $\qquad \alpha\colon A \otimes_{A^H} A \to A\#H$

$\beta\colon A \otimes_{A\#H} A \to A^H$, and

$\langle A^H, P, {}_{A^H}S_P, {}_P S_{A^H}, \alpha', \beta'\rangle$, $\qquad \alpha'\colon S \otimes_P S \to A^H$, $\quad \beta'\colon S \otimes_{A^H} S \to P$.

It is easy to see that

$$\langle A\#H, P, {}_P S \otimes_{A^H} A_{A\#H}, {}_{A\#H}A \otimes_{A^H} S_P, \gamma, \delta\rangle$$

is a Morita context, where

$$\gamma\colon A \otimes S \otimes S \otimes A \xrightarrow{1\otimes\alpha'\otimes 1} A \otimes A^H \otimes A \cong A \otimes A \xrightarrow{\alpha} A\#H,$$

$$\delta\ :\ S \otimes A \otimes A \otimes S \xrightarrow{1\otimes\beta\otimes 1} S \otimes A^H \otimes S \cong S \otimes A \xrightarrow{\beta'} P.$$

Since A/A^H is H^*-Galois, α is surjective; since $(kG)^*$ is semisimple, α' is surjective. So γ is surjective and we can apply Proposition 3.8.

Finally, the following result complements Theorem 1.3(2).

COROLLARY 3.12: *If A has an element of trace 1 and A_G is strongly graded, then* $A_G\#(kG)^* - \mathrm{Mod}$ *is equivalent with a quotient category of* $A\#H - \mathrm{Mod}$.

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